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Learning Objectives:

From this module students may get to know about the following:

1. *Spherical harmonic expansion of vector fields, particularly useful for the study of multipole radiation from localized sources.*
2. *Scalar wave equation and its spherical wave solutions.*
3. *Detailed study of spherical harmonics.*
4. *Angular momentum operator and the angular functions.*
5. *Transverse electric and transverse magnetic solutions of the electromagnetic field.*

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30 Multipole Fields - I

30.1 Scalar wave equation

Spherical harmonic expansion was extensively used in the undergraduate course for solution of problems of electrostatics, particularly boundary value problems involving certain symmetries. For time-varying phenomenon it is useful to study spherical harmonic expansion of vector fields. This is particularly useful for the study of multipole radiation from localized sources. We now look at such vector multipole fields.

Let us begin with the spherical wave solutions of the scalar wave equation before we study the vector wave equation. Let the scalar field $\psi(\vec{x},t)$ satisfy the homogeneous scalar wave equation

$$\nabla^2\psi(\vec{x},t)-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0 \tag{1}$$

We try to solve this equation by the Fourier transform method. We define the Fourier transform of $\psi(\vec{x},t)$ in time

$$\psi(\vec{x},t)=\int_{-\infty}^{\infty}\psi(\vec{x},\omega)e^{-i\omega t}d\omega \tag{2}$$

so that

$$\psi(\vec{x},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\vec{x},t)e^{i\omega t}dt \tag{3}$$

On substituting equation (2) into (1), we find that the Fourier component $\psi(\vec{x},\omega)$ satisfies the Helmholtz equation

$$(\nabla^2+\frac{\omega^2}{c^2})\psi(\vec{x},\omega)=0 \tag{4}$$

It is very often convenient to find the solution of the Helmholtz equation (or wave equation) in coordinates other than Cartesian coordinates. In particular, for systems possessing spherical symmetry about some point, which can conveniently be taken to be the origin, solution in terms of spherical coordinates is more appropriate. We use the spherical polar coordinate representation of the Laplacian operator:

$$\nabla^2=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r.)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial .}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 .}{\partial\phi^2} \tag{5}$$

where (.) represents the function on which it acts. Using this form for the Laplace operator, equation (4) becomes

$$[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r.)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial.}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2.}{\partial\phi^2}+k^2]\psi(r,\theta,\phi,\omega)=0 \qquad (6)$$

where

$$k=\frac{\omega}{c} \qquad (7)$$

30.2 Spherical Harmonics

Let us solve equation (6) by the usual method of separation of variables. In this direction write $\psi(r,\theta,\phi,\omega)$ as

$$\psi(r,\theta,\phi,\omega)=R(r)Y(\theta,\phi) \qquad (8)$$

On substituting into equation (6), we get two independent equations

$$\begin{aligned}&\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})+k^2=\lambda\\&\frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta})+\frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}=-\lambda\end{aligned} \qquad (9)$$

where λ is a constant, independent of the coordinates. The second equation can be further simplified by assuming solutions of the form

$$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi) \qquad (10)$$

Applying the method of separation of variables once again to the second of equation (9), we have

$$\begin{aligned}&\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2\\&\lambda\sin^2\theta+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})=m^2\end{aligned} \qquad (11)$$

where m is again some constant. A priori, m is a complex constant, but because Φ must be single valued, $\Phi(\phi+2n\pi)=\Phi(\phi)$, Φ must be a periodic function of ϕ, m is necessarily an integer and Φ is a linear combination of the complex exponentials $e^{\pm im\phi}$. The solution function $Y(\theta, \varphi)$ is regular at the poles of the sphere, where $\theta = 0, \pi$. Imposing this regularity in the solution Θ of the second equation at the boundary points of the domain is a Sturm–Liouville problem that forces the parameter λ to be of the form $\lambda = \ell(\ell + 1)$ for some non-negative integer with $\ell \geq |m|$. Furthermore, a change of variable to $t=\cos\theta$ transforms this equation into the Associated Legendre equation, whose solution is a multiple of the associated "Legendre polynomial", $P_l^m(\cos\theta)$. For a given value of ℓ, there are $2\ell + 1$ independent solutions of this form, one for each integer m with $-\ell \leq m \leq \ell$. These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre polynomials:

$$Y_l^m(\theta,\phi) = Ne^{im\phi}P_l^m(\cos\theta) \tag{12}$$

Several different normalizations are in common use for the Laplace spherical harmonic functions. We use the standard convention regarding associated Legendre polynomials

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x);\;\; P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \tag{13}$$

The Laplace spherical harmonics are generally defined as

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{14}$$

From equation (13) it then follows that

$$Y_l^{-m}(\theta,\phi) = (-1)^m Y_l^{m^*}(\theta,\phi) \tag{15}$$

The normalization and orthogonality conditions on the spherical harmonics are:

$$\int Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi)d\Omega = \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{16}$$

Using these spherical harmonics, the solution of the Helmholtz equation (4), or equivalently equation (6) can be written as

$$\psi(\vec{x},\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l(r)Y_l^m(\theta,\phi) \tag{17}$$

30.3The radial functions

On substituting this solution back into equation (6), we find that the radial function $f_l(r)$ satisfies the equation

$$[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}]f_l(r)=0 \qquad (18)$$

We make the substitution

$$f_l(r)=r^{-1/2}u_l(r) \qquad (19)$$

and obtain the following equation for $u_l(r)$:

$$[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(l+\frac{1}{2})^2}{r^2}]u_l(r)=0 \qquad (20)$$

Now this is nothing but the well known *Bessel equation*:

$$[\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+(1-\frac{\nu^2}{x^2})]y(x)=0 \qquad (21)$$

with the parameter $\nu=l+\frac{1}{2}$ and $x=kr$. This is thus Bessel equation of half odd-integer order.

Two solutions of the Bessel equation (21) are the *Bessel Functions of the first kind* , $J_\nu(x)$ and $J_{-\nu}(x)$. Their series expansions are:

$$J_\nu(x)=(\frac{x}{2})^\nu\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(j+\nu+1)}(\frac{x}{2})^{2j}$$
$$J_{-\nu}(x)=(\frac{x}{2})^{-\nu}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(j-\nu+1)}(\frac{x}{2})^{2j} \qquad (22)$$

If ν *is not an integer*, the two solutions are linearly independent. For integer ν the two solutions are *not linearly independent*; in fact, for *j* = *n*, an integer,

$$J_{-n}(x)=(-1)^nJ_{-n}(x) \qquad (23)$$

The two solutions that are linearly independent, whether ν is an integer or not are $J_\nu(x)$ and the *Neumann function* (or *Bessel Functions of the second kind*), defined by

$$N_\nu(x)=\frac{J_\nu(x)\cos\nu\pi-J_{-\nu}(x)}{\sin\nu\pi} \qquad (24)$$

Thus the general solution of equation (18) can be written as

$$f_l(r)=\frac{A}{r^{1/2}}J_{l+1/2}(kr)+\frac{B}{r^{1/2}}N_{l+1/2}(kr) \qquad (25)$$

For half odd-integer order, instead of the Bessel functions $J_\nu(x)$ and $N_\nu(x)$, it is more

convenient to use what are called *spherical Bessel functions* and *spherical Hankel functions*, $j_l(x), n_l(x)$ and $h_l^{(1,2)}(x)$:

$$
\begin{aligned}
j_l(x) &= \left(\frac{\pi}{2x}\right)^{1/2} J_{l+1/2}(x) \\
n_l(x) &= \left(\frac{\pi}{2x}\right)^{1/2} N_{l+1/2}(x) \\
h_l^{(1,2)}(x) &= \left(\frac{\pi}{2x}\right)^{1/2} [\, j_l(x) \pm i n_l(x)]
\end{aligned}
\tag{26}
$$

From the differential equation for the spherical Bessel functions or the generating function we can show that

$$
\begin{aligned}
j_l(x) &= (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right) \\
n_l(x) &= -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right)
\end{aligned}
\tag{27}
$$

From these formulas we can easily find explicit expressions for the spherical Bessel functions for specific values of *l*. Expressions for first few of these are

$$
\begin{aligned}
& j_0(x) = \frac{\sin x}{x}; \qquad n_0(x) = -\frac{\cos x}{x}; \qquad h_0^{(1,2)}(x) = \mp i \frac{e^{ix}}{x}; \\
& j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}; \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}; \qquad h_1^{(1,2)}(x) = \mp \frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right); \\
& j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}; \qquad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3\sin x}{x^2}; \\
& h_2^{(1,2)}(x) = \pm i \frac{e^{ix}}{x}\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right);
\end{aligned}
\tag{28}
$$

The small argument limits of the spherical Bessel functions can be obtained from equations (23), (24) and (26), and are

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$$\underline{x<<1} \qquad \begin{cases} j_l(x) \to \dfrac{x^l}{1.3.5\ldots(2l+1)}(1-\dfrac{x^2}{2(2l+3)}+\ldots); \\ n_l(x) \to -\dfrac{1.3.5\ldots(2l-1)}{x^{l+1}}(1-\dfrac{x^2}{2(1-2l)}+\ldots) \end{cases} \tag{29}$$

Similarly, for large argument, limits are

$$\underline{x>>l} \qquad \begin{cases} j_l(x) \to \dfrac{1}{x}\sin(x-l\pi/2) \\ n_l(x) \to -\dfrac{1}{x}\cos(x-l\pi/2) \\ h_l^{(1,2)}(x) \to \pm(-i)^{l+1}\dfrac{e^{\pm ix}}{x} \end{cases} \tag{30}$$

The general solution of equation (4) in spherical coordinates can be written as

$$\psi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}^{(1)}h_l^{(1)}(kr) + A_{lm}^{(2)}h_l^{(2)}(kr)]Y_l^m(\theta,\phi) \tag{31}$$

The coefficients $A_{lm}^{(1)}$ and $A_{lm}^{(2)}$ are arbitrary so far and are determined from the boundary conditions to be imposed on the solution of the wave equation.

30.4 The angular functions

So far we have concentrated on the radial functions for the scalar wave equation. We now re-look at the angular functions and introduce some concepts of use in the study of the vector wave equation. The basic angular functions are the spherical harmonics $Y_l^m(\theta,\phi)$ which are the solutions of the second of equation (9):

$$[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)]Y_l^m(\theta,\phi) = 0 \tag{32}$$

In quantum mechanics this operator is associated with orbital angular momentum. If we define the differential operator

$$\vec{L} = -i(\vec{r}\times\vec{\nabla}) \tag{33}$$

then in quantum mechanics, orbital angular momentum operator is $\hbar\vec{L}$. On using the explicit form (33) and the operator $\vec{\nabla}$ in spherical polar coordinates we can easily verify that

$$L^2 = -[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}] \quad (34)$$

where

$$L^2 = \vec{L}.\vec{L} = L_x{}^2 + L_y{}^2 + L_z{}^2 \quad (35)$$

Hence equation (32) can be written as

$$L^2 Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi) \quad (36)$$

Instead of the Cartesian components L_x, L_y, L_z it is more convenient to use the combinations

$$\begin{aligned} L_+ &= L_x + iL_y = e^{i\phi}(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}) \\ L_- &= L_x - iL_y = e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}) \\ L_z &= -i\frac{\partial}{\partial\phi} \end{aligned} \quad (37)$$

From its definition (33) it follows that for the differential operator $\vec{L}$

$$\vec{r}.\vec{L} = 0 \quad (38)$$

Further from the recursion relations for the spherical harmonics and the explicit forms (37) for ($L_\pm, L_z$), the following relations can be proved easily

$$\begin{aligned} L_\pm Y_l^m &= \sqrt{(l\mp m)(l\pm m+1)}Y_l^{m\pm1} \\ L_z Y_l^m &= mY_l^m \end{aligned} \quad (39)$$

We are in fact quite familiar with these relations from our study of orbital angular momentum in quantum mechanics. We are also familiar, from quantum mechanics, with the following easily established operator *commutation relations*

$$\begin{aligned} L^2\vec{L} &= \vec{L}L^2; \\ \vec{L}\times\vec{L} &= i\vec{L} \\ L_j\nabla^2 &= \nabla^2 L_j \end{aligned} \quad (40)$$

where ∇^2 is the Laplacian operator.

30.5 Multipole expansion of the electromagnetic field

Whatever we have done so far has been in the context of scalar fields and scalar wave equation obeyed by the scalar field. The electromagnetic fields $\vec{E}$ and $\vec{H}$ (it is more customary to deal with the magnetic field $\vec{H}$ rather than the magnetic induction field $\vec{B}$) are vector fields and we have to now develop the corresponding formalism for such fields. In vacuum the source-free Maxwell's equations are

$$\vec{\nabla}.\vec{E} = 0\,, \qquad \vec{\nabla}\times\vec{H} = \varepsilon_0 \partial\vec{E}/\partial t \qquad (41)$$

$$\vec{\nabla}\times\vec{E} + \mu_0 \frac{\partial\vec{H}}{\partial t} = 0\,, \qquad \vec{\nabla}.\vec{H} = 0 \qquad (42)$$

Assuming a time dependence of the form $e^{-i\omega t} = e^{-ickt}$ (or equivalently considering the components of the Fourier transforms in time), the equations take the form

$$\vec{\nabla}.\vec{E} = 0\,; \qquad \vec{\nabla}\times\vec{H} = -i\frac{k}{Z_0}\vec{E} \qquad (43)$$

$$\vec{\nabla}\times\vec{E} = ikZ_0\vec{H}\,; \qquad \vec{\nabla}.\vec{H} = 0 \qquad (44)$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = \mu_0 c$. On taking the curl of the second of equations (43) and using equations (44) we have

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}.\vec{H}) - \nabla^2\vec{H} = -i\frac{k}{Z_0}(\vec{\nabla}\times\vec{E}) = k^2\vec{H}$$

Or

$$(\nabla^2 + k^2)\vec{H} = 0\,, \qquad \vec{\nabla}.\vec{H} = 0 \qquad (45)$$

and

$$\vec{E} = i\frac{Z_0}{k}\vec{\nabla}\times\vec{H} \qquad (46)$$

Equivalently, on taking the curl of the first of equations (44) and using equation (43), we obtain

$$(\nabla^2 + k^2)\vec{E} = 0\,, \qquad \vec{\nabla}.\vec{E} = 0 \qquad (47)$$

and

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$$\vec{H} = -i\frac{1}{Z_0 k}\vec{\nabla}\times\vec{E} \qquad (48)$$

The set of equations (45) and (46) or (47) and (48) is equivalent to the Maxwell's equations (43) and (44).

We have already solved the wave equation in terms of a multipole expansion for the scalar field. We can follow the same procure here as well and then impose the condition $\vec{\nabla}.\vec{E} = 0$ or $\vec{\nabla}.\vec{H} = 0$ as the case may be. Or we can follow the following alternative procedure:

$$\nabla^2(\vec{r}.\vec{E}) = \vec{E}.(\nabla^2\vec{r}) + 2(\vec{\nabla}.\vec{E}) + \vec{r}.\nabla^2\vec{E}$$

But $\vec{\nabla}.\vec{E} = 0$ and $\nabla^2\vec{r} = 0$.

Hence

$$\nabla^2(\vec{r}.\vec{E}) = \vec{r}.\nabla^2\vec{E} = -k^2\vec{r}.\vec{E}$$

Therefore

$$(\nabla^2 + k^2)(\vec{r}.\vec{E}) = 0 \qquad (49)$$

Exactly similarly

$$(\nabla^2 + k^2)(\vec{r}.\vec{H}) = 0 \qquad (50)$$

In other words, both the scalars, $\vec{r}.\vec{E}$ and $\vec{r}.\vec{H}$, satisfy the scalar Helmholtz equation and equation (31) represents the general solution for both of them.

We now define a *magnetic multipole field of order* (l,m) in the following way:

$$\vec{r}.\vec{H}^{(M)}_{l,m} = \frac{l(l+1)}{k} g_l(kr) Y_l^m(\theta,\phi); \qquad \vec{r}.\vec{E}^{(M)}_{l,m} = 0\,; \qquad (51)$$

where

$$g_l(kr) = A_l^{(1)} h_l^{(1)}(kr) + A_l^{(2)} h_l^{(2)}(kr) \qquad (52)$$

Now, $\vec{r}.\vec{H}$ can be related to the electric field by using equation (48):

$$\vec{r}.\vec{H} = -i\frac{1}{Z_0 k}\vec{r}.(\vec{\nabla}\times\vec{E}) = -i\frac{1}{Z_0 k}(\vec{r}\times\vec{\nabla}).\vec{E} = \frac{1}{Z_0 k}(\vec{L}.\vec{E})$$

where we have used equation (33) to write $\vec{L}$ instead of $\vec{r}\times\vec{\nabla}$. For the magnetic multipole fields, $\vec{r}.\vec{H}$ is given by equation (51), so that

$$\vec{L}.\vec{E}^{(M)}_{l,m}(r,\theta,\phi)=l(l+1)Z_0 g_l(kr)Y_l^m(\theta,\phi); \qquad \vec{r}.\vec{E}^{(M)}_{l,m}=0 \tag{53}$$

We have been able to determine the scalar product of the electric field with $\vec{r}$ and with $\vec{L}$. To find the electric field from these, we make the following observations. The operator $\vec{L}$ acts only on the angular variables, since $\vec{L}f(r)=\frac{1}{i}(\vec{r}\times\vec{\nabla})f(r)=0$. This implies that the radial dependence of $\vec{E}$ is given by $g_l(kr)$. Secondly, from equation (39) it follows that the operator $\vec{L}$ acting on Y_l^m can at best change the value of *m*; *l* remains invariant. Thus the components of $\vec{E}$ can be linear combination of various Y_l^m for a fixed *l* which must be equal to the value of *l* on the right hand side of equation (53). Further $\vec{E}^{(M)}_{l,m}(r,\theta,\phi)$ must be such a linear combination of Y_l^m's for various *m*, such that $\vec{L}.\vec{E}^{(M)}_{l,m}(r,\theta,\phi)$ involves only a single value of *m*. $\vec{L}.\vec{L}$ is the only scalar operator under which Y_l^m's are invariant, and from equation (36) $L^2Y_l^m(\theta,\phi)=l(l+1)Y_l^m(\theta,\phi)$. Hence the electric field must be

$$\vec{E}^{(M)}_{l,m}(r,\theta,\phi)=Z_0 g_l(kr)\vec{L}Y_l^m(\theta,\phi) \tag{54}$$

This equation together with equation (48)

$$\vec{H}^{(M)}_{l,m}=-i\frac{1}{Z_0 k}\vec{\nabla}\times\vec{E}^{(M)}_{l.m} \tag{55}$$

specifies the electric and magnetic fields of the magnetic multipoles. Because these electric fields given by equation (54) are transverse to $\vec{r}$ [equation (51)], these fields are often called *transverse electric* (*TE*) rather than magnetic multipole fields.

In an identical manner we define fields of the *electric multipole* [*or transverse magnetic* (*TM*) *multipole*] *of order* (*l*,*m*) by

$$\vec{r}.\vec{E}^{E}_{l,m}=-Z_0\frac{l(l+1)}{k}f_l(kr)Y_l^m(\theta,\phi); \qquad \vec{r}.\vec{H}^{E}_{l,m}=0 \tag{56}$$

The functions $f_l(kr)$ are given by an equation similar to equation (52)

$$f_l(kr)=B_l^{(1)}h_l^{(1)}(kr)+B_l^{(2)}h_l^{(2)}(kr) \tag{57}$$

Following exactly the same procure as for the *TE* fields, we have in this case

$$\vec{H}^{(E)}_{l,m}(r,\theta,\phi)=f_l(kr)\vec{L}Y_l^m(\theta,\phi) \tag{58}$$

along with equation (46)

$$\vec{E}_{l,m}^{(E)} = i\frac{Z_0}{k}\vec{\nabla}\times\vec{H}_{l,m}^{(E)} \tag{59}$$

The two sets, magnetic multipole and electric multipole, together form a complete set of solutions to Maxwell equations in a source free region. Though the terminology, *TE* and *TM*, is often used for these fields, the terminology "magnetic multipole" and "electric multipole" is preferable, since the source of the two types of radiation, as we shall see later, are the magnetic moment density and electric charge density, respectively.

Though the spherical harmonics $Y_l^m(\theta,\phi)$ are properly normalized,

$$\int Y_{l'}^{m'*}Y_l^m d\Omega = \delta_{l'l}\delta_{m'm}$$

vector spherical harmonics, $\vec{L}Y_l^m(\theta,\phi)$ are not. It is useful to introduce the normalized form for these

$$\vec{X}_l^{\,m}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\vec{L}Y_l^m(\theta,\phi), l\neq 0 \tag{60}$$

For $l=0$, $\vec{X}_l^{\,m}(\theta,\phi)$ is defined to be identically zero. The case of $l=0$ corresponds to spherically symmetric solution and as we have already seen (radiation by simple radiating systems), such solutions exist only in the limit $k\to 0$.

The orthogonality properties of $\vec{X}_l^{\,m}(\theta,\phi)$ follow from those of $Y_l^m(\theta,\phi)$ and equations (36) to (40).

$$\int \vec{X}_{l'}^{\,m'^*}.\vec{X}_l^{\,m} d\Omega = \delta_{ll'}\delta_{mm'} \tag{61}$$

$$\int \vec{X}_{l'}^{\,m'^*}.(\vec{r}\times\vec{X}_l^{\,m})d\Omega = 0 \tag{62}$$

When both types of fields are present simultaneously, the total electric or magnetic field is the sum of two terms, expressions (55) and (58) for the magnetic and (54) and (59) for the electric fields. Thus the general solution of the Maxwell equations (43) and (44) is

$$\vec{H} = \sum_{l,m}[a_E(l,m)f_l(kr)\vec{X}_l^{\,m} - \frac{i}{k}a_M(l,m)\vec{\nabla}\times(g_l(kr)\vec{X}_l^{\,m})] \tag{63}$$

$$\vec{E} = Z_0\sum_{l,m}[a_M(l,m)g_l(kr)\vec{X}_l^{\,m} + \frac{i}{k}a_E(l,m)\vec{\nabla}\times(f_l(kr)\vec{X}_l^{\,m})] \tag{64}$$

Here the coefficients $a_E(l,m)$ and $a_M(l,m)$ specify the amounts of the two types of multipole fields. Each of $f_l(kr)$ and $g_l(kr)$ contains two constants. All these constants that appear in these equations are determined by the sources and the boundary conditions to be imposed on the fields. These can be obtained in terms of $(\vec{r}.\vec{H})$ and $(\vec{r}.\vec{E})$. From equation (63)

$$\vec{r}.\vec{H} = \sum_{l,m} a_M(l,m)\frac{\sqrt{l(l+1)}}{k} g_l(kr) Y_l^m(\theta,\phi)$$

and on using the orthonormality of spherical harmonics, equation (16),

$$\int Y_{l'}^{m'*}\vec{r}.\vec{H}d\Omega = \sum_{l,m} a_M(l,m)\frac{\sqrt{l(l+1)}}{k} g_l(kr)\int Y_{l'}^{m'*}Y_l^m d\Omega = a_M(l',m')\frac{\sqrt{l'(l'+1)}}{k} g_{l'}(kr)$$

Or

$$a_M(l,m) g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_l^{m*}(\vec{r}.\vec{H})d\Omega$$

Exactly similarly, equation (64) will yield

$$a_E(l,m) f_l(kr) = -\frac{k}{Z_0\sqrt{l(l+1)}}\int Y_l^{m*}(\vec{r}.\vec{E})d\Omega$$

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Summary

1. *In this module we have considered the multipole expansion of the electromagnetic field which is particularly useful for the study of multipole radiation from localized sources.*
2. *We first revisit the scalar wave equation and its spherical wave solutions.*
3. *Next we present a detailed study of spherical harmonics.*
4. *We then go over to the study of angular momentum operator and the angular functions, with which we are quite familiar from quantum mechanics.*
5. *Finally we obtain the transverse electric and transverse magnetic solutions of the electromagnetic field.*

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